Well-posedness for the heat flow of polyharmonic maps with rough initial data

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Abstract

We establish both local and global well-posedness of the heat flow of polyharmonic maps from $\mathbb{R}^n$ to a compact Riemannian manifold without boundary for initial data with small BMO norms.

1 Introduction

For $k \geq 1$, let N be a k-dimensional compact Riemannian manifold without boundary, isometrically embedded in some Euclidean space $\mathbb{R}^l$. For $n \geq 2$ and $m \geq 1$, we consider the m-th order energy functional

$$E_m(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla^m u|^2 = \begin{cases} \frac{1}{2}\int_{\mathbb{R}^n} |\Delta^{\frac{m}{2}} u|^2 & \text{if } m \text{ is even} \\ \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \Delta^{\frac{m-1}{2}} u|^2 & \text{if } m \text{ is odd} \end{cases}$$

for any $u \in W^{m,2}(\mathbb{R}^n, N)$, where Δ is the Laplace operator on $\mathbb{R}^n$ and

$$W^{m,2}(\mathbb{R}^n, N) = \left\{ v \in W^{m,2}(\mathbb{R}^n, \mathbb{R}^l) : \ v(x) \in N \text{ for a.e. } x \in \Omega \right\}.$$

Recall that a map $u \in W^{m,2}(\mathbb{R}^n, N)$ is called a polyharmonic map if u is the critical point of E_m. The Euler-Lagrange equation of polyharmonic maps is (see Gastel-Scheven [9]):

$$\begin{aligned} (-1)^m \Delta^m u = F(u) :=& (-1)^m \mathrm{div}^m \left(\sum_{k=0}^{m-2} \binom{m-1}{k} \nabla^{m-k-1}(\Pi(u)) \nabla^{k+1} u \right) \\ & - \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} \mathrm{div}^k \left(\nabla^{m-k}(\Pi(u)) \nabla^m u \right) \end{aligned} \tag{1.1}$$

where $\Pi : N_\delta \to N$ is the nearest point projection from the δ-neighborhood of N to N, which is smooth provide $\delta = \delta(N) > 0$ is sufficiently small. It is readily seen that (1.1) becomes the equation of harmonic maps for $m = 1$, and of extrinsic biharmonic maps for $m = 2$.

Motivated by the study of heat flow of harmonic and biharmonic maps, we consider the heat flow of polyharmonic maps, i.e. $u : \mathbb{R}^n \times \mathbb{R}_+ \to N$ solves

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$$
\begin{aligned}
u_t + (-1)^m \Delta^m u =& F(u) \quad \text{in } \mathbb{R}^n \times (0, +\infty) && (1.2)\\
u\big|_{t=0} =& u_0 \qquad \text{on } \mathbb{R}^n, && (1.3)
\end{aligned}
$$

where $u_0 : \mathbb{R}^n \to N$ is a given map.

The heat flow of harmonic maps, (1.2) for $m = 1$, has been extensively studied. For smooth initial data, the existence of global smooth heat flow of harmonic maps has been established by (i) Eells-Sampson [7] under the assumption that the sectional curvature $K_N \leq 0$, and (ii) Hildebrandt-Kaul-Widman [11] under the assumption that the image of u_0 is contained in a geodesic ball B_R in N with radius $R < \frac{\pi}{2\sqrt{\max_{B_R} |K_N|}}$. In general, the short time smooth heat flow of harmonic maps may develop singularity at finite time, see Coron-Ghidaglia [4], Chen-Ding [2], and Chang-Ding-Ye [3]. However, Chen-Struwe [6] (see also Chen-Lin [5] and Lin-Wang [19]) proved the existence of partially smooth, global weak solutions to (1.2)-(1.3) for smooth initial data u_0. For rough initial data u_0, the second author recently proved in [25] the well-posedness for the heat flow of harmonic maps provided the BMO norm of u_0 is small.

When $m = 2$, (1.2) becomes the heat flow of extrinsic biharmonic maps, which was first studied by Lamm in [15, 16, 17]. In particular, it was proven in [15, 16, 17] that if $n = 4$ and $\|u_0\|_{W^{2,2}(\mathbb{R}^4)}$ is sufficiently small, then there exists a unique global smooth solution. For an arbitrary $u_0 \in W^{m,2}(\mathbb{R}^{2m})$, it was later independently proved by Wang [23] (for $m = 2$) and Gastel [8] (for $m \geq 2$) that there exists a global weak solution to (1.2)-(1.3) that is smooth away from finitely many singular times. Very recently, the second author established in [24] the well-posedness for the heat flow of biharmonic maps for u_0 with small BMO norm.

We would like to mention that there have been some works on the regularity of polyharmonic maps for $m \geq 3$ in the critical dimensions $n = 2m$. We refer the readers to Gastel-Scheven [9], Lamm-Wang [18], Goldstein-Strzelecki-Zatorska-Goldstein[10], Moser [20], and Angelsberg-Pumberger [1].

In this paper, we are interested in the well-posedness of the heat flow of polyharmonic maps with rough initial data. In particular, we aim to extend the techniques from [25, 24] to establish the well-posedness of the heat flow of polyharmonic maps (1.2) and (1.3) for $m \geq 3$ with u_0 having small BMO norm.

We remark that the techniques employed by Wang [25, 24] were motivated by the earlier work by Koch and Tataru [14] on the global well-posedness of the incompressible Navier-Stokes equation, and the recent work by Koch-Lamm [13] on geometric flows with rough initial data.

We first recall the BMO spaces. For $x \in \mathbb{R}^n$ and $r > 0$, let $B_r(x) \subset \mathbb{R}^n$ be the ball with center x and radius r. For $f : \mathbb{R}^n \to \mathbb{R}$, let $f_{x,r}$ be the average of f over $B_r(x)$.

Definition 1.1 *For $f : \mathbb{R}^n \to \mathbb{R}$ and $R > 0$, define*

$$
\mathrm{BMO}_R(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \middle| \ [f]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n, 0 < r \leq R} r^{-n} \int_{B_r(x)} |f - f_{x,r}| < +\infty \right\}.
$$

When $R = +\infty$, we simply write $\mathrm{BMO}(\mathbb{R}^n)$ *for* $\mathrm{BMO}_\infty(\mathbb{R}^n)$.

For $0 < T \le \infty$, define the functional space X_T by

$$X_T := \left\{ f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^l \mid \|f\|_{X_T} := \sup_{0<t\le T} \|f\|_{L^\infty(\mathbb{R}^n)} + [f]_{X_T} \right\}, \tag{1.4}$$

where

$$[f]_{X_T} = \sum_{k=1}^{m} \{ \sup_{0<t\le T} t^{\frac{k}{2m}} \|\nabla^k f\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<r\le T^{\frac{1}{2m}}} (r^{-n} \int_{P_r(x,r^{2m})} |\nabla^k f|^{\frac{2m}{k}})^{\frac{k}{2m}} \} \tag{1.5}$$

where $P_r(x,r^{2m}) = B_r(x) \times [0,r^{2m}]$. It is clear that $(X_T, \|\cdot\|_{X_T})$ is a Banach space. When $T = +\infty$, we simply write X, $\|\cdot\|_X$, and $[\cdot]_X$ for X_∞, $\|\cdot\|_{X_\infty}$, and $[\cdot]_{X_\infty}$ respectively.

The main theorem is

Theorem 1.2 *There exists an $\varepsilon_0 > 0$ such that for any $R > 0$ if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \varepsilon_0$, then there exists a unique global solution $u : \mathbb{R}^n \times [0, R^{2m}] \to N$ to (1.2) and (1.3) with small semi-norm $[u]_{X_{R^{2m}}}$.*

As a direct consequence, we have

Corollary 1.3 *There exists an $\varepsilon_0 > 0$ such that if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} \le \varepsilon_0$, then there exists a unique global solution $u : \mathbb{R}^n \times \mathbb{R}_+ \to N$ to (1.2) and (1.3) with small semi-norm $[u]_X$.*

We follow the arguments in [25, 24] very closely. The paper is written as follows. In section 2, we present some basic estimates on the polyharmonic heat kernel. In section 3, we present some crucial estimates on the polyharmonic heat equation. In section 4, we prove Theorem 1.2.

2 The polyharmonic heat kernel

In this section, we will prove some basic properties on the polyharmonic heat kernel.

The fundamental solution of the polyharmonic heat equation:

$$b_t(x,t) + (-1)^m \Delta^m b(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \tag{2.1}$$

is given by

$$b(x,t) = t^{-\frac{n}{2m}} g\left(\frac{x}{t^{\frac{1}{2m}}}\right), \tag{2.2}$$

where

$$g(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi - |\xi|^{2m}} d\xi, \; x \in \mathbb{R}^n. \tag{2.3}$$

It is easy to see that g is smooth, radial, and

Proposition 2.1 *For any $L \ge 0, k \ge 0$, there exists $C = C(k,L) > 0$ such that*

$$|\nabla^k g(x)| \le C(1+|x|)^{-L}, \; \forall x \in \mathbb{R}^n. \tag{2.4}$$

Proof. For $k \geq 0$ and $L \geq 0$, since

$$\nabla_x^k \left(e^{ix\cdot\xi}\right) = (i\xi)^k (ix)^{-L} \nabla_\xi^L \left(e^{ix\cdot\xi}\right),$$

we have, by integration by parts,

$$\begin{aligned} |\nabla^k g(x)| &= \left|\int_{\mathbb{R}^n} (ix)^{-L} e^{ix\cdot\xi} \nabla_\xi^L \left((i\xi)^k e^{-|\xi|^{2m}}\right) d\xi\right| \\ &\leq C(k,L)(1+|x|)^{-L}. \end{aligned}$$

This completes the proof. $\square$

As a direct consequence of (2.4), we have the following properties for the polyharmonic heat kernel b

Lemma 2.2 *For any $k, L \geq 0$, there exist $C_1 > 0$ depending on n, L and $C_2, C_3 > 0$ depending on n, k, L such that for any $x \in \mathbb{R}^n$ and $t > 0$, it holds:*

$$|b(x,t)| \leq C_1 t^{-\frac{n}{2m}} \left(1 + \frac{|x|}{t^{\frac{1}{2m}}}\right)^{-L}, \tag{2.5}$$

$$|\nabla^k b(x,t)| \leq C_2 \left(t^{-\frac{1}{2m}}\right)^{n+k-L} \left(t^{\frac{1}{2m}} + |x|\right)^{-L}, \tag{2.6}$$

$$\|\nabla^k b(x,t)\|_{L^1(\mathbb{R}^n)} \leq C_3 t^{-\frac{k}{2m}}. \tag{2.7}$$

At the end of this section, we recall that the solution to the Dirichlet problem of inhomogeneous polyharmonic heat equation

$$u_t(x,t) + (-1)^m \Delta^m u(x,t) = f(x,t) \text{ in } \mathbb{R}^n \times \mathbb{R}_+, \tag{2.8}$$

$$u(x,0) = u_0(x) \text{ on } \mathbb{R}^n \tag{2.9}$$

is given by the following Duhamel formula:

$$u = \mathbf{G}u_0 + \mathbf{S}f, \tag{2.10}$$

where

$$\mathbf{G}u_0(x,t) := \int_{\mathbb{R}^n} b(x-y,t)u_0(y)dy, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{2.11}$$

and

$$\mathbf{S}f(x,t) := \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)f(y,s)dyds, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{2.12}$$

3 Basic estimates for the polyharmonic heat equation

In this section, we will provide some crucial estimates for the solution of the polyharmonic heat equation with initial data in BMO spaces.

Lemma 3.1 *For $0 < R \leq +\infty$, if $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$, then $\hat{u}_0 := \mathbf{G}u_0$ satisfies*

$$\sum_{k=1}^{m} \sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_{P_r(x,r^{2m})} r^{2k-2m} |\nabla^k \hat{u}_0|^2 \leq C\,[u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)}\,, \tag{3.1}$$

and

$$\sum_{k=1}^{m} \sup_{0<t\leq R^{2m}} t^{\frac{k}{2m}} \left\| \nabla^k \hat{u}_0(t) \right\|_{L^\infty(\mathbb{R}^n)} \leq C\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}\,. \tag{3.2}$$

If, in addition, $u_0 \in L^\infty(\mathbb{R}^n)$ then

$$\sum_{k=1}^{m-1} \sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_{P_r(x,r^{2m})} |\nabla^k \hat{u}_0|^{\frac{2m}{k}} \leq C\,\|u_0\|^{\frac{2m}{k}-2}_{L^\infty(\mathbb{R}^n)} \cdot [u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)}\,, \tag{3.3}$$

The proof of Lemma 3.1 is similar to [24] Lemma 3.1. For completeness, we sketch it here. Let $\mathcal{S}$ denote the class of Schwartz functions, the following characterization of BMO spaces, due to Carleson, is well-known (see, Stein [21]).

Lemma 3.2 *For $0 < R \leq +\infty$, let $\Phi \in \mathcal{S}$ be such that $\int_{\mathbb{R}^n} \Phi = 0$ and denote for $t > 0$, $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, $x \in \mathbb{R}^n$. If $f \in \mathrm{BMO}_R(\mathbb{R}^n)$, then*

$$\sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_0^r \int_{B_r(x)} |\Phi_t * f|^2(x,t) \frac{dxdt}{t} \leq C\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \tag{3.4}$$

for some $C = C(n) > 0$.

Proof of Lemma 3.1. Let g be given by (2.3) and $\Phi^i = \nabla^i g$ for $i = 1, \cdots, m$. Then $\Phi^i \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \Phi^i = 0$ for $i = 1, \cdots, m$. Direct calculations show

$$\Phi^i_t(x) = t^{-n}(\nabla^i g)\left(\frac{x}{t}\right) = t^i \nabla^i \left(t^{-n} g(\frac{x}{t})\right) = t^i \nabla^i g_t(x),$$

where $g_t(x) = t^{-n} g(\frac{x}{t})$. Hence we have

$$\Phi^i_t * u_0(x) = t^i \nabla^i \left(g_t * u_0\right)(x).$$

Since the polyharmonic heat kernel $b(x,t) = g_{t^{\frac{1}{2m}}}(x)$, we have

$$\Phi^i_t * u_0(x) = t^i \nabla^i [(b(\cdot, t^{2m}) * u_0)(x)] = t^i \nabla^i (\mathbf{G}u_0)(x, t^{2m}).$$

Thus Lemma 3.1 implies that for $i = 1, \cdots, m$,

$$\begin{aligned} C\,[u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)} \geq & \sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_0^r \int_{B_r(x)} |\Phi^i_t * u_0|^2 \frac{dxdt}{t} \\ = & \sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_0^r \int_{B_r(x)} t^{2i-1} |\nabla^i \mathbf{G}u_0|^2(x, t^{2m}) dxdt \\ = & \frac{1}{2m} \sup_{x\in\mathbb{R}^n, 0<r\leq R} r^{-n} \int_{P_r(x,r^{2m})} t^{\frac{2i-2m}{2m}} |\nabla^i \mathbf{G}u_0|^2(x,t) dxdt. \end{aligned}$$

This clearly implies (3.1), since for $i = 1, \cdots, m$, $t^{\frac{2i-2m}{2m}} \geq r^{2i-2m}$ when $0 < t \leq r^{2m}$.

Since $\hat{u}_0$ solves the polyharmonic heat equation:

$$(\partial_t + (-1)^m \Delta^m)\hat{u}_0 = 0 \text{ on } \mathbb{R}^n \times (0, +\infty),$$

the standard theory implies that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\sum_{k=1}^{m} r^{\frac{k}{m}} |\nabla^k \hat{u}_0|^2(x, r^{2m}) \leq C \sum_{k=1}^{m} r^{-n} \int_{P_r(x,r^{2m})} r^{2k-2m} |\nabla^k \hat{u}_0|^2.$$

Taking supremum over $x \in \mathbb{R}^n$ and $0 < t = r^{2m} \leq R^{2m}$ yield (3.2).

For (3.3), observe that $u_0 \in L^\infty(\mathbb{R}^n)$ implies $\Phi_t^i * u_0 \in L^\infty(\mathbb{R}^n)$ for $i = 1, \cdots, m-1$, and

$$\|\Phi_t^i * u_0\|_{L^\infty(\mathbb{R}^n)} \leq \|\Phi^i\|_{L^1(\mathbb{R}^n)} \|u_0\|_{L^\infty(\mathbb{R}^n)} \leq \|\nabla^i g\|_{L^1(\mathbb{R}^n)} \|u_0\|_{L^\infty(\mathbb{R}^n)} \leq C \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Hence

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^n, 0 < r \leq R} r^{-n} \int_{P_r(x,r^{2m})} |\nabla^i \mathbf{G} u_0|^{\frac{2m}{i}} dxdt \\
&= \sup_{x \in \mathbb{R}^n, 0 < r \leq R} r^{-n} \int_{B_r(x) \times [0,r]} |\Phi_t^i * u_0|^{\frac{2m}{i}} \frac{dxdt}{t} \\
&\leq \left(\sup_{t > 0} \|\Phi_t^i * u_0\|_{L^\infty(\mathbb{R}^n)} \right)_{L^\infty(\mathbb{R}^n)}^{\frac{2m}{i} - 2} \cdot \sup_{x \in \mathbb{R}^n, 0 < r \leq R} r^{-n} \int_{B_r(x) \times [0,r]} |\Phi_t^i * u_0|^2 \frac{dxdt}{t} \\
&\leq C \|u_0\|_{L^\infty(\mathbb{R}^n)}^{\frac{2m}{i} - 2} \cdot [u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)}
\end{aligned}$$

This implies (3.3). $\square$

Now we prove an important estimate on the distance of $\hat{u}_0$ to the manifold N in term of the BMO norms of u_0. More precisely,

Lemma 3.3 *For any $\delta > 0$, there exists $K_0 = K_0(\delta, N) > 0$ such that for $0 < R \leq +\infty$, if $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$ then*

$$dist(\hat{u}_0(x,t), N) \leq K_0 [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \delta, \ \forall x \in \mathbb{R}^n, \ 0 \leq t \leq (\frac{R}{K_0})^{2m}. \tag{3.5}$$

Proof. For any $x \in \mathbb{R}^n$, $t > 0$ and $K > 0$, denote

$$c_{x,t}^K = \frac{1}{|B_K(0)|} \int_{B_K(0)} u_0(x - t^{\frac{1}{2m}} z) dz.$$

Let g be given by (2.3). Then, by a change of variables, we have

$$\hat{u}_0(x,t) = \int_{\mathbb{R}^n} g(y) u_0(x - t^{\frac{1}{2m}} y) dy.$$

Applying (2.5) (with $L = n+1$) from Lemma 2.2, we have

$$
\begin{aligned}
\left|\hat{u}_0(x,t) - c^K_{x,t}\right| &\le \int_{\mathbb{R}^n} g(y) \left|u_0(x - t^{\frac{1}{2m}} y) - c^K_{x,t}\right| dy \\
&\le \left\{ \int_{B_K(0)} + \int_{\mathbb{R}^n \setminus B_K(0)} \right\} g(y) \left|u_0(x - t^{\frac{1}{2m}} y) - c^K_{x,t}\right| dy \\
&\le \int_{B_K(0)} \left|u_0(x - t^{\frac{1}{2m}} y) - c^K_{x,t}\right| dy \\
&\quad + C\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_K(0)} \frac{1}{|y|^{n+1}}\, dy \\
&\le K^n \left[u_0\right]_{\mathrm{BMO}_{Kt^{\frac{1}{2m}}}(\mathbb{R}^n)} + \delta,
\end{aligned}
\tag{3.6}
$$

provided we choose a sufficiently large $K = K_0(\delta, N) > 0$ so that

$$
C\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_K(0)} \frac{1}{|y|^{n+1}}\, dy \le \delta.
$$

On the other hand, since $u_0(\mathbb{R}^n) \subset N$, we have

$$
\mathrm{dist}(c^K_{x,t}, N) \le \frac{1}{|B_K(0)|} \int_{B_K(0)} \left|c^K_{x,t} - u_0(x - t^{\frac{1}{2m}} y)\right| dy \le [u_0]_{\mathrm{BMO}_{Kt^{\frac{1}{2m}}}(\mathbb{R}^n)} \cdot \tag{3.7}
$$

Putting (3.6) and (3.7) together yields (3.5) holds for $t \le (\frac{R}{K})^{2m}$. □

4 Boundedness of the operator S

In this section, we introduce several function spaces and establish the boundedness of the operator $\mathbf{S}$ between these spaces.

For $0 < T < \infty$, the spaces Y^k_T, for $k = 0, \cdots, m-1$, are the sets consisting of all functions $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ such that

$$
\|f\|_{Y^k_T} := \sup_{0<t\le T} t^{\frac{2m-k}{2m}} \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<R\le T^{\frac{1}{2m}}} \left(R^{-n} \int_{P_R(x)} |f|^{\frac{2m}{2m-k}} \right)^{\frac{2m-k}{2m}}. \tag{4.1}
$$

Notice that $(Y^k_T, \|\cdot\|_{Y^k_T})$ is a Banach space for $k = 0, \cdots, m-1$. When $T = +\infty$, we simply denote $(Y^k, \|\cdot\|_{Y^k})$ for $(Y^k_\infty, \|\cdot\|_{Y^k_\infty})$.

Let the operator $\mathbf{S}$ be defined by (2.12). Then we have

Lemma 4.1 *For any $0 < T \le +\infty$ and $k = 0, \cdots, m-1$, if $f \in Y^k_T$, then $\mathbf{S}(\nabla^\alpha f) \in X_T$ and*

$$
\|\mathbf{S}(\nabla^\alpha f)\|_{X_T} \le C\, \|f\|_{Y^k_T}\,, \tag{4.2}
$$

where $\alpha = (\alpha_1, \cdots \alpha_n)$ is any multi-index of order k.

Proof. We need to show the point wise estimate

$$\sum_{i=0}^{m} R^i|\nabla^i \mathbf{S}(\nabla^\alpha f)|(x,R^{2m}) \le C\|f\|_{Y_T^k}, \quad \forall x\in\mathbb{R}^n,\ 0<R\le T^{\frac{1}{2m}}, \tag{4.3}$$

and the integral estimate for $0<R\le T^{\frac{1}{2m}}$:

$$\sum_{i=1}^{m} R^{-\frac{in}{2m}} \left\|\nabla^i \mathbf{S}(\nabla^\alpha f)\right\|_{L^{\frac{2m}{i}}(P_R(x,R^{2m}))} \le C\left\|f\right\|_{Y_T^k} \tag{4.4}$$

By suitable scaling, we may assume $T\ge 1$. Since both estimates are translation and scale invariant, it suffices to show (4.3) and (4.4) hold for $x=0$ and $R=1$.

For $i=0,\cdots,m$ and $\alpha=(\alpha_1,\cdots,\alpha_n)$ with order k, we have

$$\begin{aligned}
\left|\nabla^i \mathbf{S}(\nabla^\alpha f)\right|(0,1) =& \left|\int_0^1\int_{\mathbb{R}^n}\nabla^{i+\alpha}b(y,1-s)f(y,s)dyds\right| \\
\le& \left\{\int_{\frac12}^1\int_{\mathbb{R}^n} + \int_0^{\frac12}\int_{B_2} + \int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}\right\}\left|\nabla^{i+k}b(y,1-s)\right|\left|f(y,s)\right|dyds \\
=& I_1+I_2+I_3.
\end{aligned}$$

Applying Lemma 2.2, we can estimate I_1, I_2 and I_3 as follows.

$$\begin{aligned}
|I_1| \le& \left(\sup_{\frac12\le s\le 1}\|f(s)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{\frac12}^1\left\|\nabla^{i+k}b(\cdot,1-s)\right\|_{L^1(\mathbb{R}^n)}ds\right) \\
\le& C\|f\|_{Y_1^k}\int_0^{\frac12}s^{-\frac{i+k}{2m}}ds \quad (\text{by } (2.7)) \\
\le& C\|f\|_{Y_1^k} \quad (\text{since } i+k\le 2m-1).
\end{aligned}$$

$$\begin{aligned}
|I_2| \le& \left(\sup_{0\le s\le\frac12}\|\nabla^{i+k}b(\cdot,1-s)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{B_2\times[0,\frac12]}|f(y,s)|dyds\right) \\
\le& C\int_{B_2\times[0,\frac12]}|f(y,s)|dyds \\
\le& C\|f\|_{Y_1^k}.
\end{aligned}$$

$$\begin{aligned}
|I_3| \le& \int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}\left|\nabla^{i+k}b(y,1-s)\right||f(y,s)|dyds \\
\le& C\int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}|y|^{-(n+1)}|f(y,s)|dyds \quad (\text{by } (2.6) \text{ for } L=n+1) \\
\le& \left(\sum_{k=2}^{\infty}k^{n-1}k^{-(n+1)}\right)\left(\sup_{x\in\mathbb{R}^n}\int_{P_1(x,1)}|f(y,s)|dyds\right) \\
\le& C\left(\sum_{k=2}^{\infty}k^{-2}\right)\|f\|_{Y_1^k}\le\|f\|_{Y_1^k}.
\end{aligned}$$

Now we want to show (4.4) by the energy method. Denote $w = \mathbf{S}(\nabla^\alpha f)$. Then w solves

$$(\partial_t + (-1)^m \Delta^m) w = \nabla^\alpha f \quad \text{in } \mathbb{R}^n \times (0, +\infty); \quad w|_{t=0} = 0. \tag{4.5}$$

Let $\eta \in C_0^\infty(B_2)$ be a cut-off function of B_1. Multiplying (4.5) by $\eta^4 w$ and integrating over $\mathbb{R}^n \times [0,1]$, we obtain

$$\int_{\mathbb{R}\times\{1\}} |w|^2 \eta^4 + 2\int_{\mathbb{R}^n\times[0,1]} \nabla^m w \cdot \nabla^m (w\eta^4) = 2\int_{\mathbb{R}^n\times[0,1]} \nabla^\alpha f \cdot w\eta^4. \tag{4.6}$$

By the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times[0,1]} \nabla^m w \cdot \nabla^m (w\eta^4) \\
=&\int_{\mathbb{R}^n\times[0,1]} |\nabla^m (w\eta^2)|^2 + \int_{\mathbb{R}^n\times[0,1]} \nabla^m w \left(\sum_{\beta=0}^{m-1} \nabla^\beta (w\eta^2) \cdot \nabla^{m-\beta}(\eta^2) \right) \\
&- \int_{\mathbb{R}^n\times[0,1]} \nabla^m (w\eta^2) \left(\sum_{\beta=0}^{m-1} \nabla^\beta (w) \cdot \nabla^{m-\beta}(\eta^2) \right) \\
\geq& \frac{1}{2}\int_{\mathbb{R}^n\times[0,1]} |\nabla^m (w\eta^2)|^2 - C\sum_{\beta=0}^{m-1} \int_{B_2\times[0,1]} |\nabla^\beta w|^2
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n\times[0,1]} \nabla^\alpha f \cdot w\eta^4 = (-1)^k \int_{\mathbb{R}^n\times[0,1]} f \cdot \nabla^\alpha [(w\eta^2)\eta^2] \\
\leq& C\sum_{\beta=0}^{k} \int_{\mathbb{R}^n\times[0,1]} |f||\nabla^\beta (w\eta^2)| \\
\leq& C\sum_{\beta=0}^{k-1} \sup_{0<t\leq 1} t^{\frac{2m-k}{2m}} \|f\|_{L^\infty(\mathbb{R}^n)} \cdot \sup_{0<t\leq 1} t^{\frac{\beta}{2m}} \|\nabla^\beta w\|_{L^\infty(\mathbb{R}^n)} \cdot \int_0^1 t^{-1+\frac{k-\beta}{2m}} dt \\
&+ C\|f\|_{L^{\frac{2m}{2m-k}}(B_2\times[0,1])} \cdot \|\nabla^k (w\eta^2)\|_{L^{\frac{2m}{k}}(\mathbb{R}^n\times[0,1])} \\
\leq& C\|f\|_{Y_1^k}^2 + C\|f\|_{Y_1^k} \cdot \|\nabla^k (w\eta^2)\|_{L^{\frac{2m}{k}}(\mathbb{R}^n\times[0,1])}
\end{aligned} \tag{4.8}$$

To estimate the last term, we need the Nirenberg interpolation inequality: for $k \leq m-1$,

$$\|\nabla^k (w\eta^2)\|_{L^{\frac{2m}{k}}(\mathbb{R}^n)}^{\frac{2m}{k}} \leq C\|w\eta^2\|_{L^\infty(\mathbb{R}^n)}^{\frac{2m}{k}-2} \|\nabla^m (w\eta^2)\|_{L^2(\mathbb{R}^n)}^2,$$

which, after integrating with respect to $t \in [0,1]$, implies

$$\|\nabla^k (w\eta^2)\|_{L^{\frac{2m}{k}}(\mathbb{R}^n\times[0,1])} \leq C \sup_{0\leq t\leq 1} \|w\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{k}{m}} \|\nabla^m (w\eta^2)\|_{L^2(\mathbb{R}^n\times[0,1])}^{\frac{k}{m}}, \tag{4.9}$$

Putting (4.9), (4.7) and (4.8) into (4.6), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n\times[0,1]}|\nabla^m(w\eta^2)|^2\\
\le& C\sum_{\beta=0}^{m-1}\int_{B_2\times[0,1]}|\nabla^\beta w|^2+C\|f\|_{Y_1^k}^2+C\|f\|_{Y_1^k}\cdot\|\nabla^k(w\eta^2)\|_{L^{\frac{2m}{k}}(\mathbb{R}^n\times[0,1])}\\
\le& C\sum_{\beta=0}^{m-1}\left[\int_0^1 t^{-\frac{\beta}{m}}dt\cdot\sup_{0<t\le1}\left(t^{\frac{\beta}{m}}\|\nabla^\beta w(t)\|_{L^\infty(\mathbb{R}^n)}^2\right)\right]\\
&+C\|f\|_{Y_1^k}^2+C\|f\|_{Y_1^k}\sup_{0\le t\le 1}\|w\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{k}{m}}\|\nabla^m(w\eta^2)\|_{L^2(\mathbb{R}^n\times[0,1])}^{\frac{k}{m}}\\
\le& C\|f\|_{Y_1^k}^2+\frac12\int_{\mathbb{R}^n\times[0,1]}|\nabla^m(w\eta^2)|^2+C\|f\|_{Y_1^k}^q\cdot\|w\|_{L^\infty(\mathbb{R}^n)}^{(1-\frac{k}{m})q}\\
\le&\frac12\int_{\mathbb{R}^n\times[0,1]}|\nabla^m(w\eta^2)|^2+C\|f\|_{Y_1^k}^2,
\end{aligned}
\tag{4.10}
$$

where $q=\frac{2m}{2m-k}$. Therefore, we obtain

$$
\int_{P_1(0,1)}|\nabla^m w|^2\le\int_{\mathbb{R}^n\times[0,1]}|\nabla^m(w\eta^2)|^2\le C\|f\|_{Y_1^k}^2. \tag{4.11}
$$

For $i=1,\cdots,m-1$, applying Nirenberg's interpolation inequality gives

$$
\begin{aligned}
&\int_{P_1(0,1)}|\nabla^i w|^{\frac{2m}{i}}\le\int_{\mathbb{R}^n\times[0,1]}|\nabla^i(w\eta^2)|^{\frac{2m}{i}}\\
\le&\sup_{0\le t\le1}\|w\|_{L^\infty(\mathbb{R}^n)}^{\frac{2m}{i}-2}\|\nabla^m(w\eta^2)\|_{L^2(\mathbb{R}^n\times[0,1])}^2\\
\le& C\|f\|_{Y_1^k}^{\frac{2m}{i}-2}\cdot\int_{\mathbb{R}^n\times[0,1]}|\nabla^m(w\eta^2)|^2\\
\le& C\|f\|_{Y_1^k}^{\frac{2m}{i}}
\end{aligned}
\tag{4.12}
$$

(4.11) and (4.12) imply (4.4). This completes the proof. □

5 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The idea is based on the fixed point theorem in a small ball inside $X_{R^{2m}}$.

Since the image of a map $u\in X_{R^{2m}}$ may not be contained in N, we first need to extend Π to $\mathbb{R}^l$, denoted as $\widetilde{\Pi}$, such that $\widetilde{\Pi}\in C^\infty(\mathbb{R}^l)$ and $\widetilde{\Pi}\equiv\Pi$ in N_{δ_N}.

Let

$$
\begin{aligned}
\widetilde{F}(u):=&(-1)^m\mathrm{div}^m\left(\sum_{k=0}^{m-2}\binom{m-1}{k}\nabla^{m-k-1}(\widetilde{\Pi}(u))\nabla^{k+1}u\right)\\
&-\sum_{k=0}^{m-1}(-1)^k\binom{m}{k}\mathrm{div}^k\left(\nabla^{m-k}(\widetilde{\Pi}(u))\nabla^m u\right).
\end{aligned}
\tag{5.1}
$$

For $k = 0, \cdots, m-2$, define

$$F_k(u) = (-1)^{k+1} \binom{m}{k} \nabla^{m-k}(\widetilde{\Pi}(u))\nabla^m u.$$

For $k = m-1$, define

$$\begin{aligned}
& F_{m-1}(u) \\
= \ & (-1)^m \binom{m}{m-1} \nabla(\widetilde{\Pi}(u))\nabla^m u \\
+ \ & (-1)^m \sum_{k=0}^{m-2} \binom{m-1}{k} \left(\operatorname{div}(\nabla^{m-k-1}(\widetilde{\Pi}(u)))\nabla^{k+1}u + \nabla^{m-k-1}(\widetilde{\Pi}(u))\operatorname{div}(\nabla^{k+1}u) \right).
\end{aligned}$$

Then (1.2) can be written as

$$(\partial_t + (-1)^m \Delta^m)u = \sum_{k=0}^{m-1} \operatorname{div}^k \left(F_k(u)\right). \tag{5.2}$$

The first observation is

Lemma 5.1 *For $0 < R \leq +\infty$, if $u \in X_{R^{2m}}$, then $F_k(u) \in Y^k_{R^{2m}}$ for $k = 0, \cdots, m-1$. Moreover, there exists $C > 0$ depends on N and $\|u\|_{L^\infty(\mathbb{R}^n \times [0,R^{2m}])}$ such that*

$$\|F_k(u)\|_{Y^k_{R^{2m}}} \leq C \sum_{l=1}^{m} [u]_{X_{R^{2m}}}^{\frac{2m-k}{l}}, \ 0 \leq k \leq m-1. \tag{5.3}$$

Proof. It follows from direct calculations and Hölder inequality that

$$|F_k(u)| \leq C \sum_{l=1}^{m} |\nabla^l u|^{\frac{2m-k}{l}}, \ 0 \leq k \leq m-1,$$

where $C > 0$ depends on N and $\|u\|_{L^\infty(\mathbb{R}^n \times [0,R^{2m}])}$. Thus we have, by the definitions of $X_{R^{2m}}$ and $Y^k_{R^{2m}}$, that

$$\|F_k(u)\|_{Y^k_{R^{2m}}} \leq C \sum_{l=1}^{m} [u]_{X_{R^{2m}}}^{\frac{2m-k}{l}}, \ \forall 0 \leq k \leq m-1.$$

This completes the proof. □

By the Duhamel formula (2.10), the solution u to (5.2) is given by

$$u(x,t) = \mathbf{G}u_0 + \sum_{k=0}^{m-1} \mathbf{S}\left(\operatorname{div}^k(F_k(u))\right). \tag{5.4}$$

From now on,denote

$$\hat{u}_0 = \mathbf{G}u_0.$$

Define the mapping operator $\mathbf{T}$ on $X_{R^{2m}}$ by letting

$$\mathbf{T}u(x,t) = \hat{u}_0 + \sum_{k=0}^{m-1} \mathbf{S}\left(\operatorname{div}^k(F_k(u))\right). \tag{5.5}$$

The following property follows directly from Lemma 3.1.

Lemma 5.2 *For any $0 < R \le +\infty$, if $u_0 : \mathbb{R}^n \to N$, then $\hat{u}_0 \in X_{R^{2m}}$, and*

$$\|\hat{u}_0\|_{L^\infty(\mathbb{R}^n\times[0,R^{2m}])} \le C\|u_0\|_{L^\infty(\mathbb{R}^n)}, \quad [\hat{u}_0]_{X_{R^{2m}}} \le C\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}\,. \tag{5.6}$$

For any $\varepsilon > 0$ and $0 < R \le +\infty$, let

$$\mathbf{B}_\varepsilon(\hat{u}_0) := \left\{ u \in X : \ \|u - \hat{u}_0\|_{X_{R^{2m}}} \le \varepsilon \right\}$$

be the ball in $X_{R^{2m}}$ with center $\hat{u}_0$ and radius ε. By the triangle inequality, we have

$$\|u\|_{L^\infty(\mathbb{R}^n\times[0,R^{2m}])} \le C\|u_0\|_{L^\infty(\mathbb{R}^n)} + \varepsilon,\ [u]_{X_{R^{2m}}} \le C\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \varepsilon,\ \forall u \in \mathbf{B}_\varepsilon(\hat{u}_0). \tag{5.7}$$

Thus we have

Lemma 5.3 *For $0 < R \le +\infty$, if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \varepsilon$, then*

$$\|u\|_{L^\infty(\mathbb{R}^{n+1}_+)} \le C + \varepsilon, \quad [u]_X \le C\varepsilon, \quad \forall u \in \mathbf{B}_\varepsilon(\hat{u}_0), \tag{5.8}$$

for some $C = C(n, N) > 0$.

The proof of Theorem 1.2 is based on the following two lemmas.

Lemma 5.4 *There exists $\varepsilon_1 > 0$ such that for $0 < R \le +\infty$, if $u_0 : \mathbb{R}^n \to N$ has*

$$[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \varepsilon_1,$$

then $\mathbf{T}$ maps $\mathbf{B}_{\varepsilon_1}(\hat{u}_0)$ to $\mathbf{B}_{\varepsilon_1}(\hat{u}_0)$.

Proof. By (5.5), we have

$$\mathbf{T}(u) - \hat{u}_0 = \sum_{k=0}^{m-1} \mathbf{S}\left(\mathrm{div}^k(F_k(u))\right), \quad u \in \mathbf{B}_{\varepsilon_1}(\hat{u}_0).$$

Hence Lemma 4.1, Lemma 5.1 and Lemma 5.2 imply that for any $u \in \mathbf{B}_{\varepsilon_1}(\hat{u}_0)$,

$$\begin{aligned}
\|\mathbf{T}(u) - \hat{u}_0\|_{X_{R^{2m}}} &\lesssim \sum_{k=0}^{m-1} \left\|\mathbf{S}(\mathrm{div}^k(F_k(u)))\right\|_{X_{R^{2m}}} \\
&\lesssim \sum_{k=0}^{m-1} \|F_k(u)\|_{Y^k_{R^{2m}}} \\
&\lesssim \sum_{k=0}^{m-1} \sum_{l=1}^{m} [u]_{X_{R^{2m}}}^{\frac{2m-k}{l}} \\
&\le C\,[u]_{X_{R^{2m}}}^{\frac{m+1}{m}} \le \varepsilon_1,
\end{aligned}$$

provide $\varepsilon_1 > 0$ is chosen to be sufficiently small. Hence $\mathbf{T}u \in \mathbf{B}_{\varepsilon_1}(\hat{u}_0)$. This completes the proof. □

Lemma 5.5 *There exist $0 < \varepsilon_2 \le \varepsilon_1$ and $\theta_0 \in (0,1)$ such that for $0 < R \le +\infty$, if $u_0 : \mathbb{R}^n \to N$ satisfies*

$$[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \varepsilon_2,$$

then $\mathbf{T} : \mathbf{B}_{\varepsilon_2}(\hat{u}_0) :\to \mathbf{B}_{\varepsilon_2}(\hat{u}_0)$ is a θ_0-contraction map, i.e.

$$\|\mathbf{T}(u) - \mathbf{T}(v)\|_{X_{R^{2m}}} \le \theta_0 \|u - v\|_{X_{R^{2m}}}, \quad \forall u, v \in \mathbf{B}_{\varepsilon_2}(\hat{u}_0).$$

Proof. For $u, v \in \mathbf{B}_{\varepsilon_2}(\hat{u}_0)$, we have

$$\begin{aligned}\|\mathbf{T}(u) - \mathbf{T}(v)\|_{X_{R^{2m}}} \le & \sum_{k=0}^{m-1} \left\| \mathbf{S}(\mathrm{div}^k(F_k(u) - F_k(v))) \right\|_{X_{R^{2m}}} \\ \lesssim & \sum_{k=0}^{m-1} \|F_k(u) - F_k(v)\|_{Y^k_{R^{2m}}} .\end{aligned} \tag{5.9}$$

Notice that $\|u\|_{L^\infty(\mathbb{R}^n \times [0,R^{2m}])} + \|v\|_{L^\infty(\mathbb{R}^n \times [0,R^{2m}])} \le C_0$. For any $k = 0, \cdots, m-2$, it follows from the definition of $F_k(u)$ we have

$$\begin{aligned}&|F_k(u) - F_k(v)| \\ \lesssim & |\nabla^m(u-v)| \left[|\nabla^{m-k} u| + \sum_{j=1}^{m-k} \left(\sum_{|\alpha|=j} (\Pi_{i=1}^n |\nabla^{\alpha_i} u|) \right) |\nabla^{m-k-j} u| \right] \\ & + |\nabla^m v||u-v| \left[\sum_{|\alpha|=m-k} (\Pi_{i=1}^n |\nabla^{\alpha_i} u| + \Pi_{i=1}^n |\nabla^{\alpha_i} v|) \right] \\ & + |\nabla^m v| \left[\sum_{j=1}^{m-k} |\nabla^j (u-v)| \left(\sum_{|\alpha|=m-k-j} (\Pi_{i=1}^n |\nabla^{\alpha_i} u| + \Pi_{i=1}^n |\nabla^{\alpha_i} v|) \right) \right].\end{aligned}$$

Hence we have

$$\begin{aligned}\|F_k(u) - F_k(v)\|_{Y^k_{R^{2m}}} \lesssim & \left[\sum_{j=1}^{m-k} ([u]^j_{X_{R^{2m}}} + [v]^j_{X_{R^{2m}}}) \right] \|u - v\|_{X_{R^{2m}}} \\ \le & C\varepsilon_2 \|u - v\|_{X_{R^{2m}}},\end{aligned} \tag{5.10}$$

where we have used Lemma 5.3 in the last step.

For $k = m-1$, since

$$\begin{aligned}|F_{m-1}(u) - F_{m-1}(v) \lesssim & [|\nabla u||\nabla^m(u-v)| + |\nabla(u-v)||\nabla^m v|] \\ & + \sum_{k=0}^{m-2} \left[|\nabla^{m-k}(\widetilde{\Pi}(u))| + |\nabla^{m-k}(\widetilde{\Pi}(v))| \right] \left| \nabla^{k+1}(u-v) \right| \\ & + \sum_{k=0}^{m-2} \left| \nabla^{m-k}(\widetilde{\Pi}(u) - \widetilde{\Pi}(v)) \right| \left[|\nabla^{k+1} u| + |\nabla^{k+1} v| \right],\end{aligned}$$

we also have

$$\|F_{m-1}(u) - F_{m-1}(v)\|_{Y^{m-1}_{R^{2m}}} \leq C\varepsilon_2 \|u-v\|_{X_{R^{2m}}}. \tag{5.11}$$

Putting (5.10) and (5.11) into (5.9), we obtain

$$\|\mathbf{T}(u) - \mathbf{T}(v)\|_{X_{R^{2m}}} \leq C\varepsilon_2 \|u-v\|_{X_{R^{2m}}} \leq \theta_0 \|u-v\|_{X_{R^{2m}}}$$

for some $\theta_0 = \theta_0(\varepsilon_2) \in (0,1)$, provided $\varepsilon_2 > 0$ is chosen to be sufficiently small. This completes the proof of Lemma 5.5. □

Proof of Theorem 1.2. It follows from Lemma 5.4 and Lemma 5.5, and the fixed point theorem that there exists $\varepsilon_0 >$ such that for $0 < R \leq +\infty$, if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \varepsilon_0$, then there exists a unique $u \in X_{R^{2m}}$ such that

$$u = \hat{u}_0 + \mathbf{S}(\widetilde{F}(u)) \quad \text{on } \mathbb{R}^n \times [0, R^{2m}],$$

or equivalently

$$u_t + (-1)^m \Delta^m u = \widetilde{F}(u) \quad \text{on } \mathbb{R}^n \times (0, R^{2m}];\ u\big|_{t=0} = u_0.$$

We want to show $u(\mathbb{R}^n \times [0, R^{2m}]) \subset N$. By Lemma 3.3, we have that for any $x \in \mathbb{R}^n$ and $0 \leq t \leq (\frac{R}{K_0})^{2m}$,

$$\begin{aligned} \mathrm{dist}(u(x,t), N) \leq& \mathrm{dist}(\hat{u}_0, N) + \|u - \hat{u}_0\|_{L^\infty(\mathbb{R}^n \times [0,R^{2m}])} \\ \leq& \delta + K_0 [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \varepsilon_0 \\ \leq& \delta + (1+K_0)\varepsilon_0 \leq \delta_N, \end{aligned}$$

provided $\delta \leq \frac{\delta_N}{2}$ and $\varepsilon_0 = \frac{\delta_N}{2(1+K_0)}$. This yields $u(\mathbb{R}^n \times [0, (\frac{R}{K_0})^{2m}]) \subset N_{\delta_N}$. Hence

$$\widetilde{\Pi}(u) = \Pi(u), \quad \widetilde{F}(u) = F(u) \text{ on } \mathbb{R}^n \times [0, (\frac{R}{K_0})^{2m}].$$

Set $Q(u) = y - \Pi(y)$ for $y \in N_{\delta_N}$, and $\rho(u) = \frac{1}{2}|Q(u)|^2$. Then direct calculations imply that for any $y \in N_{\delta_N}$,

$$\begin{aligned} \nabla Q(y)(v) &= (\mathrm{Id} - \nabla \Pi(y))(v),\ \forall v \in \mathbb{R}^l \\ \nabla^2 Q(y)(v,w) &= -\nabla^2 \Pi(y)(v,w),\ \forall v, w \in \mathbb{R}^l. \end{aligned}$$

Set $A(y)(v,w) = -\nabla^2 \Pi(y)(v,w)$ for $y \in N_{\delta_N}$ and $v, w \in \mathbb{R}^l$. Then $F(u)$ can be rewritten by (see Gastel [8]):

$$F(u) = (-1)^{m+1} \sum_{j=0}^{2m-2} \sum_{|\alpha|=2m-2-j} \binom{2m-2-j}{\alpha} \mathrm{tr}^m (\nabla^j A) \circ u (\nabla^{\alpha_1+1} u, \nabla^{\alpha_2+1} u, \cdots, \nabla^{\alpha_{j+2}+1} u).$$

Direct calculations imply

$$
\begin{aligned}
&\Delta^m Q(u) = \nabla Q(u)(\Delta^m u)\\
&+\sum_{j=0}^{2m-2}\sum_{|\alpha|=2m-2-j}\binom{2m-2-j}{\alpha}\mathrm{trace}^m(\nabla^j A)\circ u(\nabla^{\alpha_1+1}u,\nabla^{\alpha_2+1}u,\cdots,\nabla^{\alpha_{j+2}+1}u)\\
&=\nabla Q(u)(\Delta^m u)+(-1)^{m+1}F(u).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
(\partial_t+(-1)^m\Delta^m)Q(u) =&[\nabla Q(u)F(u)-F(u)]\\
=&-\nabla\Pi(u)(F(u)).
\end{aligned} \tag{5.12}
$$

Multiplying both sides of (5.12) by $Q(u)$ and integrating over $\mathbb{R}^n$, we obtain that for $0\le t\le (\frac{R}{K_0})^{2m}$,

$$
\frac{d}{dt}\int_{\mathbb{R}^n}\rho(u)+\int_{\mathbb{R}^n}|\nabla^m Q(u)|^2=-\int_{\mathbb{R}^n}\langle\nabla\Pi_(u)(F(u)),Q(u)\rangle=0, \tag{5.13}
$$

where we have used the fact that $Q(u)\perp T_{\Pi(u)}N$ and $\nabla\Pi(u)(F(u))\in T_{\Pi(u)}N$ on $\mathbb{R}^n\times[0,(\frac{R}{K_0})^{2m}]$ in the last step.

Since $\rho(u)|_{t=0}=0$, integrating (5.13) with respect to t implies $\rho(u)\equiv 0$ on $\mathbb{R}^n\times[0,(\frac{R}{K_0})^{2m}]$. Thus $u(\mathbb{R}^n\times[0,(\frac{R}{K_0})^{2m}])\subset N$. Repeating the same argument also implies that $u(\mathbb{R}^n\times[(\frac{R}{K_0})^{2m},R^{2m}])\subset N$. This completes the proof of Theorem 1.2. $\square$

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